THE

MATHEMATICAL GAZETTE

A TERMINAL JOURNAL FOR STUDENTS AND TEACHERS.

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ALGEBRA IN SCHOOLS.

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I AM venturing to call the attention of the Association to a difficulty that I have to contend with, as a private tutor, when assisting boys fresh from school either to continue their mathematical studies or to prepare for some of the public examinations. It is of course impossible for me to assert positively that this difficulty is experienced by tutors generally, but the strong conviction that it is felt by others as well as by myself has led to my bringing the matter forward.

In the case of the great majority of schoolboys their mathematical outfit includes some knowledge of arithmetic, algebra, Euclid, and trigonometry. In the case of three of these, the arithmetic, Euclid, and trigonometry, the tutor finds a certain amount of knowledge gained, which he will probably have to strengthen and make more accurate, but which will in some degree serve as a foundation. In algebra, however, the work has usually to be done all over again. The reason for this appears to be that there is an ever-growing divergence between the conception of the nature and objects of algebra that dominates school teaching and the conception that regulates the application of algebra to more advanced mathematics. As a rough attempt to give, in as few words as possible, the character of this divergence, it may be said that at school the end in view is the solution of equations, while later on the student finds his main business is the study of functions.

A teacher who makes it his endeavour to treat the subject in a more satisfactory way has, fortunately, the advantage of the guidance of eminent authorities. Chrystal's *Algebra* and Clifford's *Common Sense of the Exact Sciences* will help him to lay out the best road, and in the published addresses of Chrystal, Henrici, Sylvester, and others there are many directions and suggestions that will be of the greatest service to him. Nevertheless, these high authorities have not as yet had much influence on school books. I do not know of any English book suitable for elementary teaching that is constructed according to their recommendations. A "Chrystal for Beginners" is sadly wanted. The *Indian Text-Book of Algebra* by Rhadakrishna, in the library of the Association, partly supplies this want, and I have myself used it with pupils with satisfactory results, but hitherto all negotiations for an English edition have broken down, and it would be necessary to send to Madras for a copy.

This persistence of school books in the old tracks is doubtless due to the magnitude of the mass to be moved before progress can be made. We sometimes, I think, forget how comparatively modern is the introduction of algebra into Europe. When it first came into England it was called the Cossick art, or the art of Cossick numbers, and Cossick was from "cosa," a thing to be found. For a long period, more than a hundred years, the notion that Professor Chrystal ascribes to his pupils that "higher algebra is the solution of harder and harder equations," was the notion gravely entertained by the best mathematicians. It was through the effort to solve harder and harder equations that all the earlier advances were made, and school teaching in this, as well as in many subjects other than mathematical, has but stood still while the boundaries of knowledge were being enlarged.

I disclaim at the outset any right to speak on mathematical subjects with authority, and the only plea on which I ask for your consideration of some suggestions as to the form which algebraical elementary teaching should take is, that whether right or wrong, they are at least based on practical experience. To save time, and to make my meaning clear, they may perhaps seem to be brought forward in too decided a manner, but I fully recognise that they have no value at all until, and so far as, they are confirmed by others of attainments superior to my own.

Algebra being described as a generalisation of arithmetic, the similarities and differences between the algebraical $f(x)$ and the arithmetical $f(t)$, where t stands for 10, should be fully explained. In the first instance, $f(x)$ should have numerical coefficients only. It would be necessary here to give the student a new view of the symbol 0, as being not merely a negation of magnitude or a starting-point of the

positive scale, but as a definite point in a scale extending infinitely on each side of it. Nothing should be defined as that which is less than any assignable quantity, and from the fact that it is indifferent which sign we ascribe to it, the corresponding indifference in the sign of $\frac{1}{0}$ or ∞ should be mentioned as something to be more fully treated later on, and might, I think, with advantage be made the subject of some geometrical illustrations. Clifford's *Common Sense of the Exact Sciences* affords a model for the treatment of the associative, commutative, and index laws, and when once these have been mastered, the pupil may be told that he may deal with the fundamental processes of algebra just as if they were processes in arithmetic, except that there is no carrying.

In this first chapter, however, on addition, subtraction, multiplication, and division as applied to an integral function of x with numerical coefficients, there is room for some things that are almost always overlooked or omitted. The student should be encouraged to work everything with detached coefficients, and he should be trained to verify his results by substituting 1, 2, or 3 for x. A habit of constant verification cannot be too soon encouraged, and the earlier it is acquired the more swiftly and almost automatically it is practised.

In the second chapter we might have $f(x)$ still integral and finite, but with literal coefficients; the theorem often spoken of as the remainder theorem, that if $f(x)$ be divided by $(x-a)$ the remainder is $f(a)$, may then be introduced; and from the corollary that if $f(a)=0$, $f(x)$ is divisible by $x-a$, the subject of factorisation is made plain and easy. There may be some doubt whether or not at this stage to introduce the principle of indeterminate coefficients, though it might easily be done. I believe it would be wise to do so, as it would afford the means of treating more completely the separation of quadratic functions into linear factors. This separation, which is the same thing as the solution of quadratic equations, would necessitate the use of the radical symbol. Following Chrystal, Todhunter, Hall and Knight, and the majority of writers, $\sqrt{a}$ should be considered a quantity having one and not two values, although the algebra of C. Smith and the article by Professor Kelland in the *Encyclopædia Britannica* make $\sqrt{a}$ have two values. Supposing some work done to give the pupil readiness in multiplying and dividing functions with literal coefficients, he might be ready to pass on to chapter three.

The third chapter would require some knowledge of the first and most elementary results in permutations and combinations. It should be noticed that Clifford in the work so often referred to starts from these at the very beginning, while Chrystal defers the subject to a very late period. These first results, being purely arithmetical in character, may be put in anywhere, and I am inclined to think the sooner the better. The student might now proceed from $f(x)$ to $f(xy)$, $f(xyz)$, and so on, noting the number and kinds of terms he may reckon on finding in a function of any of these classes of given degree. The consideration of these expressions and their classification would afford ample material for explaining the terms "dimensions," "homogeneous," and "symmetry." At present, although "homogeneous" is usually defined somewhere in the first three pages of a school algebra, the schoolboy never knows anything about its meaning, as he has not been used to apply it. Symmetry should then be explained and freely illustrated, the symbols Σ and Π used by Chrystal for a product should be introduced. If, for example, *xyz*, *abc* were the elements, such symmetrical products as $\Pi(x+y-c)$ and $\Pi(x+y-a-b)$ should be expanded, the result being expressed with the Σ and Π symbols, the pupil being taught strictly to account for the twenty-seven terms of the first or the sixty-four terms of the second. The permutations and combinations that were mentioned just now will be wanted for the consideration of the product $(x+a)(x+b)(x+c)$ . . . etc.; and the binomial theorem for a positive integral index may be then deduced.

If the next chapter deals with fractions, and the pupil has been taught to work Least Common Multiple in arithmetic by separation into factors and not by the division method, and has been shown the special characteristics of the Highest Common Factor process in algebra, I think he may be told that every process in algebra is step by step identical with the corresponding process in arithmetic, with one exception, namely, that it is often better to add algebraic fractions first two together, then a third, and so on, instead of all at once. The subjects of ratio, proportion, and variation are all fractions over again, with some new notation and new names added, several of the symbols being purely alternative, some, as for instance, $a:b:c:d=p:q:r:s$, expressing in a concise form what could have previously been stated only in some lengthy fashion.

I will not trespass on your time longer by carrying the subject on further. It is not the treatment of indices, surds, the complete binomial theorem and logarithms that makes the tutor's task a difficult one. It is because he finds that what ought to have been learnt at the beginning has never been learnt, and that his pupils have got into a clumsy mechanical way of dealing with questions that it is almost impossible to get them out of. No doubt this difficulty has arisen from hurrying over the rudiments so as to arrive at problems in equations, in order that boys may have some plausible answer to their natural question, "What is the good of algebra?" and because it is believed that these problems interest the youthful mind. I sometimes feel a doubt, however, whether boys really enjoy being introduced to such exercises as, "A says to B, how much money have you got?" and B makes a very singular hypothetical reply; or to the fish whose body is half as long again as his head and tail together, while head and tail have given relations of magnitude. I cannot but

suspect that they must feel that there is something unpractical in these problems. There must, however, be equations, and there should be some illustrative problems, but the other parts of algebra should not be sacrificed to them. So long as the classification of functions, the remainder theorem, and the analogies and differences between arithmetic and algebra are all duly attended to, the fish, the market-woman with her eggs, the stage-coaches in opposite directions, and other old friends may be occasionally welcome to enliven the scene.

Let me conclude by again earnestly asking you to believe that I am not making these recommendations in any spirit of presumption. While thoroughly convinced that there is something wrong in our school system, I am equally convinced of the great improbability of my having hit on the best remedy, but have still a hope that even crude and imperfect suggestions may point the way to something that will be far better. G. HEPPEL.

MATHEMATICS FOR ASTRONOMY AND NAVIGATION.

I was led to the present subject by observing that every one, almost without exception, is interested in astronomy, yet from lack of learning spherical trigonometry very few of those who have passed through an ordinary school and college course of mathematics can understand a book of physical astronomy, or at least any other than a very elementary and popular exposition of this royal science. The ordinary well-educated man cannot only not predict an eclipse, but he has no idea of the method followed in the prediction. His astronomy is all taken on faith or on authority; he knows only results, nothing of processes. The work of Kepler is as much a mystery to him as a theological dogma. If he possess a telescope, he can use it little better than a toy; he has no power of adding anything to the sum of astronomical knowledge. Now this seems a pity: I think it is not necessary, for my experience tells me that spherical trigonometry is not much harder than plane trigonometry.

I make bold to say that an ordinary well-educated man has less notions of physical astronomy than a lady who has been taught at her boarding school the use of the globes. Take, *e.g.*, this problem from Keith, p. 351, Prob. lxxvi: *The lat. of a place and the day of the month being given, to find all those stars that rise and set acronycally and heliacally.* I confess my own inability to answer it without the book, even were I allowed the globe. By the bye, should not a pair of globes be an article of teaching apparatus in every school? It is only on a globe that the relative form and position of countries can be accurately represented. Two angles, lat. and long.; and two signs, N. and S., and E. and W. (given the centre and the radius), would serve to represent the dimensions of a sphere far better than length, breadth, and thickness; and it is probable that if mankind had earlier known that they were inhabitants of a globe, geometry would have been very different from what it is now. How much would a little knowledge of spherics simplify the doctrine of poles and polars and reciprocation. Is it not a real disgrace how few men or women have the slightest knowledge of how to draw the curved lines on any map; or what is meant by conical, orthogonal, or Mercator's projection; or how to ascertain and record their own position if they happen to be travelling outside the region of maps, clocks, and telegraphs; or how to make a sun-dial; or how to tell the time from the position of sun, moon or star; or how to set a wind-vane; or which way the variation of the compass needle is now changing in England or on the Atlantic? A very clever man who had spent most of his life road-surveying in New Zealand told me that *the sun was always due E. or W. at six o'clock!* How many know what the *Saros* is, though known to the Greek astronomer; or the relation between a sphere and a cylinder of equal height and diameter, discovered by Archimedes; or how eclipses are calculated, though to explain this principle requires no mathematics higher than algebra; or whether eclipses of the sun or moon are most frequent? How many could tell that the planets are to be looked for in the zodiac, but not the asteroids? How many understand anything about precession of the equinoxes, though known before the Christian era; or from what point R. A. is reckoned; or where the first point in Aries is? How many know that a ship's shortest course is along a great circle; or how long. and lat. are discovered by the sextant?

I believe there are very few references to plane trigonometry in Casey's *Spherical Trigonometry*, which I suppose is at least as good as any other hand-book.

I would say experience teaches me never to ask for definitions, unless the pupil has been bidden (which he seldom should be) to repeat a definition, and never to flatly contradict a pupil's answer if you can avoid it, but if it is vague and uncertain, try and get him by questioning and suggesting to improve it into what is clear and sufficient. As to the question of *names* v. *things*, I think the modern preference for things may be carried too far. Mr. Ruskin says truly, I think, that the main thing learnt even at a university is the power of expression. It may be said we must have something to express; but there is not likely to be any lack of that when we have to teach the elements or write even business letters. A pupil may be excellent in the manipulation of algebraic formulæ, but from his non-cultivation of language very deficient in power of expressing even a simple geometrical truth in ordinary language.

To teach spherical trigonometry or spherics, I take it we should have a large wooden globe painted a dull black and marked with a faint equator and meridians. On this arcs of great and small circles should be drawn with a pair of compasses provided with white chalk pencil. To draw a great circle easily I suppose one leg at least should be

bent or bendable, so as to be like the legs of a pair of calipers. Cones of paper, etc. should be provided to illustrate the projections. An orange will do for minor illustrating, a black ink line being easily seen on its yellow skin.

As to the vexed question of the innateness of mathematical ideas *v.* their derivation from experience, can we go farther than to say that the capacity is innate?—all definitions must be derived from experience. I suppose the mind of blind men differs little from that of a seeing person *ab origine*. The blind have the *capacity* for distinguishing colours: if their eyes are couched, they begin to exercise it. Who ever conceived clearly a mathematical line who had not first seen the pencil or ink trace that represents it; or an infinite plane who had not seen and thought about a thin sheet of paper; or the flat surface of a sandy plain; or of still water?

Most young pupils are most easily interested in *applications* of mathematical formulæ, and I believe that many if not most of the recent advances of pure mathematics have been due to exigencies of the arts. In chemistry, in strength of materials, in physics, in electricity, in theory of music, problems are continually occurring which call for new developments of mathematics. But for dynamics and astronomy should we ever have had a differential calculus?

In conclusion, let no one despair that he is too old for mathematics. Classics claim their favourites and too often their victims young, but *sera nunquam est ad mathesim via*, dull eyes may late find a life of interest in objects animate and inanimate, and dull ears may be late awakened to the music of the spheres.

T. WILSON.

A SECOND CHAPTER ON CONICS.

1. Let PQ, PQ′ (Fig. 1) be tangents to a conic, S, H the foci, Q′C*q* a diameter, then K*q* the tangent at *q* is ‖PQ′. By a construction and a proof identical with those of an article in the second number of this *Gazette* we can show that rect^{le} PQ . QK = SQ . QH, $\angle$ SKQ = QHP = PHQ′, Δ^s PKS and PHQ′ similar, and $\therefore$ rect^{le} PS . PH = PK . PQ′. Through C draw a straight line ‖QQ′, meeting PQ, PQ′ in F, F′, and cutting the tangent *q*K in E. QQ′ being ‖FF′ and bisected by CP, CF = CF′; also Q′*q* being a diameter, CE = CF′. Thus E and F are the same point and coincide with K, and $\therefore$ KC is ‖QQ′.

Hence PN : PC = PQ : PK = PQ . PQ′ : PK . PQ′,
= PQ . PQ′ : PS . PH,

i.e. "the rectangle contained by the tangents from P : that contained by the focal distances of P = PN : PC."

2. Now let ABC (Fig. 2) be any Δ^{le} inscribed in a circle; draw the median line AN and the symmedian chord AU, making with AB the same angle that AN makes with AC. From similar Δ^s AUB, ACN we have rect^{le} AB . AC = AU . AN. Then draw the diameter AO and the perpendicular AM. These also are known to make equal angles with the sides, and we infer that $\angle$ UAO = MAN = compt of ANB.

3. Next describe a circle (Fig. 3) through the foci and any given external point P, and let it meet in U a diameter CU which makes with the major axis the same angle that CP makes with it, and produce PC to meet the circle in W. Then C being the middle point of the chord SH, and CU, CW equally inclined to that chord, CU = CW, and UW is parallel to SH, and consequently arc SU = arc HW and $\angle$ SPU = HPW, $\therefore$ by the last Art.

rect^{le} PS . PH = UP . PC.

But we have shown PS . PH : PQ . PQ′ = PC : PN, $\therefore$ = UP . PC : UP . PN,

$\therefore$ **PQ . PQ′ = UP . PN.**

Moreover, PS, PH make equal angles with the tangents, and we have just shown that $\angle$ UPS = WPH, $\therefore$ the whole $\angle$ UPQ = NPQ′, $\therefore$ Δ^s UPQ, NPQ′ are similar, and $\angle$ PUQ = PQ′Q, $\therefore$ U lies on the circle PQQ′.

Thus "The circles PQQ′ and PSH cross each other at a point U lying on a diameter CU which makes with the major axis the same angle that CP makes with it. Also rect^{le} CP . CU = CP . CW = CS . CH = CS^2, $\therefore$ the angles PCS and SCU are similar."

To save repetition it will be convenient to call CU the diameter *contrary* to CP.

4. From the above investigation follow certain deductions, old and new.

α. When the conic becomes a parabola, PSU is a straight line bisected at S. If a name is wanted for the point U, it may be called the *ultra-focal* point or *ultra* point of P.

β. Let normals at Q, Q′ meet in O, then PO being a diameter of the circle PQQ′, $\angle$ UPO which is the complement of PNQ is constant while P moves along any fixed diameter, $\therefore$ the right angled Δ UPO is of constant species. In the case of a parabola, PU is bisected at S, $\therefore$ SUO is also a triangle of constant species. One corner of it is fixed at S, another U moves along a straight line parallel to that on which P moves, $\therefore$ the third corner O also moves along a fixed straight line. This gives us the important proposition, "Normals at the ends of parallel chords of a parabola meet on a fixed straight line."

γ. Given the centre C of a conic, two points Q, Q′ on the curve, and the tangent at one of them Q, to find the foci. Draw CP bisecting the chord QQ′ and meeting the given tangent in P; then PQ′ is a tangent. Describe the circle PQQ′, and find the point U by drawing the symmedian chord from P. The major axis bisects the $\angle$ PCU, and the relation CP . CU = CS^2 gives the foci.

δ. Draw the radius CR parallel to the tangent PQ. Since PQ . QK = CR^2 we have

PN : NC = PQ : QK = PQ^2 : PQ . QK = PQ^2 : CR^2;

and similarly for CR′ the radius ‖PQ′.

Hence it follows that **CR . CR′ = rect^{le} UP . NC**; and from that comes the proposition, "The rect^{le} contained by the focal distances of P is equal to the sum of the rect^{le} con-

tained by the tangents from P, and the rect^{le} contained by the radii parallel to those tangents."

ϵ. Let the circle PQQ′ cut PC in V, then arc UQ = VQ′, and UV is ‖QQ′.

If a be the radius on which P lies and β the conjugate radius, we may assume that CN . CP = a^2 and NV. CP = β^2. Thus CV. CP = $a^2 - \beta^2$. If $a > \beta$, C lies outside the circle PQQ′. If $a < \beta$, C is within it. If $a = \beta$, *i.e.* if P lies on one of the equi-conjugate diameters, the circle passes through C, and as $\angle$ PCO is then a right angle we have the proposition, "Normals at the ends of chords parallel to one of the equi-conjugate diameters meet on the diameter at right angles to the other equi-conjugate."

ζ. Let tangents be drawn from any fixed point P to a series of confocal conics; then CU is fixed in direction, and as CS^2 is constant, the position of U is fixed. Hence "in this case all such circles as PQQ′ have a common chord PU, and the corresponding normals meet on a straight line through U perpendicular to PU."

5. Three other deductions are left for the reader: the first is easy, the others a little harder.

α. Let CP meet the conic in R. From R draw RG, RG′ to the major axis ‖ the focal distances of P. Prove the rect^{le} RG . RG′ = rect^{le} contained by the radii ‖ the tangents from P.

β. The axes of all conics which touch two given straight lines at given points are tangents to a fixed parabola (U is the focus and PN the directrix).

γ. From U draw any straight line meeting the circle PQQ′ in M, and QQ′ produced in R. Join PM. Show that the angles at M = those at N, and that PM is the polar of the point R with regard to any conic touching PQ, PQ′ at Q, Q′.

A particular case of this proposition, viz., when P is on the major axis, and $\therefore$ the $\angle^s$ at N, M are right angles, and QU, Q′U are normals, is familiar enough.

E. P. Rouse.

SOME OLD TEXT-BOOKS.

(i.) *The Young Mathematician's Guide.* By John Ward (Eighth Edition, 1747, London).

This product of Queen Anne's reign consists of five parts: I. Arithmetick, Vulgar and Decimal. II. Algebra or Arithmetick in Species. III. The Elements of Geometry. IV. Conick Sections. V. The Arithmetick of Infinites explained, with its application to Superficial and Solid Geometry. To these is added an Appendix of Practical Gauging, and in the eighth edition is also found the History of Logarithms.

The dedication is inscribed to Sir Richard Grosvenor of Eaton, and is followed by the Preface "to the Reader," which sets forth the exceeding usefulness of those foundations of mathematical learning—Arithmetick and Geometry.

The first part of the work, that on arithmetic, has very little out of the common, save some curious words and phrases. Of course the clumsy terms for the higher powers, such as biquadrat, sursolid, square cubed, second sursolid, and biquadrat squared, are used. As Ward states in his Preface, the rule of false or rule of position is left out, "because all such Questions as can be answered by that guessing rule are much better done by any one who hath but a very small smattering of Algebra." Thus he shows himself in advance of many of his time. For example, James Dodson in his 19th edition of Wingate's *Arithmetic* inserts this cumbrous rule in 1760, so does Hill in his Arithmetic of 1722. But of course his system, since it brings in algebra, allows him this loophole which is not allowed to such writers as make no mention of algebra. Simple proportion is treated in a rather cumbersome way; three different ways of working out a simple proportion, when the *statement* has been made, are given. The first consists in dividing the product of the second and third terms by the first term; the second and third consist in dividing the first into the second or third term and multiplying by the other. Some need of compression and consolidation is here evident. But the words which bear on the making of the *statement* are worth recording: "When three numbers are given, and it is required to find a fourth Proportional, the greatest Difficulty (if there be any) will be in the right stating the Question, or abstracting the Numbers out of the Words in the Question, and placing them down in their proper order. Now this will be very easy if it be truly considered that always two of the three given Terms are only supposed and assigned, or limit the Ratio or Proportion. The third moves the Question, and the fourth gives the Answer."

It may be noted that the relation between Troy and Avoirdupois was not quite settled, for our author says: "I found by a very nice Experiment that one pound Averdupois is equal to 14 ounces, 11 pennyweights, and 15½ grains Troy." This is nearly, but not exactly, the modern value, for it is equivalent to 6999½ grains. Again, with regard to specific gravity, he prides himself on some tables he has made himself, and "which I can assure you required more Charge, Care, and Trouble than I was at first aware of"; but he gives also the figures of Henry van Etten (1633), of Alsted (1649), of the "ingenious Mr. Oughtred (1660), following one Marinus Ghetaldi," and of numbers 169 and 199 of the Philosophical Transactions.

Algebra, to which are also given the names algebraick arithmetick, algebra in specie, logistica speciosa or specious computation (Mr. Harriot), begins with an explanation of signs and symbols, wherein it is of note that the unknowns or "numbers to be sought" are represented by small vowels, and the given quantities are represented by small consonants. There are some noticeable words and signs. *Uncia*, so common last century, appears in the discussion of the Binomial Theorem (the which theorem Mr. Ward fancied he had discovered for himself and the world, but honestly owns that he found he had been anticipated by

Sir Isaac Newton). The "first or leading letter" is used for that letter by which expressions are arranged, and here I find that I myself have been anticipated by Mr. Ward. For in an Algebra by a friend of mine is a footnote, at my suggestion, that "leading quantity" might be a good phrase for such a letter or quantity. Two signs for involution and evolution ($\mathcal{C}$ and $\mathcal{uq}$) have not survived. *Fellow-factor* is a better expression than co-factor as an alternative for coefficient.

In the algebra itself is an attempt to prove the "rule of signs in multiplication" by a method like this: if $4a - 7b = 0$, therefore $4a = 7b$, therefore $12af = 21bf$, therefore $12\ af - 21b = 0$. In G. C. M. there is no introduction of simple factors to help the work, but if divisor began $3x^2$ and dividend $4x^3$ the quotient would run $\frac{4}{3}x$. Interest is ranged under the head of algebra; and both simple and compound interest are worked by formulæ. In a scholium is pointed out the fact that in computing half a year's interest (compound interest being used) a man should pay $10\frac{1}{2}$d. less than £103 for a loan of £100 at 6 per cent. It is pleasing to come across the phrase, "However, it is most like an Artist to perform Things of this Nature the nearest and easiest Way they can be done." It is also pleasing to find appended to an article on Mortality Tables according to Sir William Petty and Dr. Halley a "serious observation," viz., "How unjustly we repine at the Shortness of our *Lives*, and think ourselves wrong'd if we attain not to *Old Age;* whereas it appears that *One-Half* of those that are BORN die in seventeen years time. . . . So that instead of murmuring at what we call a *Short Life*, we ought to account it as a great Blessing that we have *survived*, perhaps by many *Years*, that *Period* of *Life* whereat the one half of the whole Race of Mankind does not arrive."

The latter half of the book is taken up with geometry, conic sections, and "the arithmetick of infinites applied to superficies and solids" (what we should call mensuration), followed by appendices on practical gauging, the construction of logarithms, and plane trigonometry. The last of these is disposed of in less than six pages, and in this short space are given rules for solution of triangles, both angled and scalene.

In geometry the proofs of the proposition widely differ both in order and method from Euclid's. The proofs of I. 27, 28, 29 are roughly thus: Bring the upper point of intersection down along the intersecting line to the lower point of intersection. Then exterior will coincide with interior, and alternate angles become vertically opposite. III. 35, 36 are proved by VI. 4. A geometrical solution for quadratics is given. At the end is a new and easy method for periphery and area of circle, and a *facile* way of making natural sines and tangents.

The portion devoted to conics is full of details, notes the degeneration of one sort of conic into another, and gives geometrical methods for drawing the different conics. A hit is made at slide-rules, "but too much practis'd, which at best do but help to guess at the truth," in practical gauging. In the tract on logarithms our author states his opinion that the three great improvements in the machinery of mathematics were (1) Indian characters, (2) decimal point, (3) logarithms. His list of log. tables comprise those of "Neper," Briggs, Vlacq, Gellibrand, Roe (who shows the best disposition), and Sherwin.

J. H. HOOKER.

NOTES.

7. On the compound pendulum.

Part II. of the Syllabus of *Elementary Dynamics*, drawn up by a committee of the A. I. G. T., will shortly be published by Messrs. Macmillan and Co. Part I. dealt with linear dynamics, and Part II. deals with co-planar dynamics.

The object of the present note is to amplify § 89*a* of Part II., which runs thus: "Compound pendulum, equivalent simple pendulum, centre of oscillation or percussion."

As regards the centre of percussion, or, more correctly, the line of percussion, that should be read after § 93 of the Syllabus.

Let the body be suspended by a horizontal axis through A (Fig. 4), perpendicular to the plane of the paper which contains the centre of gravity (G) of the body, and which is called the plane of oscillation. Let a be the distance AG between the axis and G, and let MK^2 be the moment of inertia about an axis through G parallel to the axis of suspension. Then the moment of inertia about the axis itself is $M(k^2 + a^2)$ [§ 89], and the energy of the body when moving with the angular velocity ω is $\frac{1}{2}(k^2 + a^2)\omega^2$ (§§ 87, 88).

Equivalent simple pendulum.—This is a heavy particle suspended by a weightless string of such length (l) that it will oscillate in the same time as the compound pendulum.

The energy and work equation of the compound pendulum is $\frac{1}{2}(k^2 + a^2)\omega^2 = ga\ (\cos\theta - \cos\alpha)$, where θ is the angle made at any moment by AG with the vertical, and α is the maximum value of θ, and the equation of the simple pendulum is $\frac{1}{2}l^2\omega^2 = gl\ (\cos\theta - \cos\alpha)$.

Hence the angular velocities of the two pendulums at every instant will be the same, and therefore also their times of oscillation will be equal, if

$$\frac{k^2 + a^2}{l^2} = \frac{a}{l}, \text{ i.e. if } l = \frac{k^2 + a^2}{a}.$$

This therefore is the length of the equivalent simple pendulum.

Construction for l.—Draw $GK = k$ perpendicular to AG, and draw KO perpendicular to AK to meet AG, produced, in O. Then

$$AO = \frac{AK^2}{AG} = \frac{a^2 + k^2}{a}, \quad \therefore\ AO = l.$$

If $OG = b$, then $k^2 = ab$, and $l = a + b$.

Centre of oscillation.—The point O is called the centre of oscillation of the compound pendulum. The time of

oscillation of the body about a parallel axis through O is the same as that about the given axis, for in each case AO is the length of the equivalent simple pendulum.

Line of percussion.—If the body, initially at rest, be struck by a blow P along a line through O perpendicular to AO and in the plane of oscillation, there will be no shock felt at A, that is, the whole of the impulse will be spent in producing motion. For, taking moments about A, we have (§ 93) $P(a+b) = M(k^2+a^2)\omega = M(ab+a^2)\omega$, therefore $P = Ma\omega$, which is the momentum of the body supposed collected at G.

If an impulse had been applied in any other manner there would have been an induced impulse on the axis such that the resultant impulse would be in magnitude and line of action equal to the above given impulse P.

Hence the line of action of P is called the *line of percussion*. The mass of the equivalent simple pendulum which would be equally moved by the impulse P is $\frac{Ma}{a+b}$.

For, call its mass M_0, then $P = M_0(a+b)\omega$, therefore

$$Ma\omega = M_0(a+b)\omega, \quad \therefore \quad M_0 = \frac{Ma}{a+b}.$$

Similarly, if O were the centre of suspension, the mass of the equivalent simple pendulum at A is $\frac{Mb}{a+b}$.

Therefore we have the following important theorem:—

> The compound pendulum moves exactly as if it were condensed into two heavy particles rigidly connected by a massless framework, one particle at A $\left(\text{of mass } \frac{Mb}{a+b}\right)$ and the other at O $\left(\text{of mass } \frac{Ma}{a+b}\right)$, the two together being equal to M, and having G for their centre of gravity.

The controlling equation between a and b is $ab = k^2$.

If any blow be struck along any line in the plane of oscillation, its effect can be graphically resolved (as in Fig. 5) into two components, one along the line of percussion producing motion, and the other having its line of action through A, and causing an impulse on the axis.

Let the line of action of the blow be from C to B, meeting the line of percussion in B, then the lines of action of the components acting at A and O are AB and OB. If the triangle of impulses be completed, as ABC, the *magnitudes* of the components can be measured. A. LODGE.

If the line of action of the blow is parallel to the line of percussion, the components at A and O can be obtained by the method of moments. A. LODGE.

8. On the harmonic polar of a conic.

Let O be the fixed point, TL the polar of O, Q′OQL any chord through O. Draw the diameter COR. Also draw the tangent RT′ (which is parallel to TL), and RW an ordinate of CP, the diameter which bisects QQ′.

Then from similar Δ^s LVT, RWT′,

$$VL : VT = RW : WT'$$

and $VO : CV = RW : CW$ from CVO, CWR,

$$\therefore \ VO \cdot VL : CV \cdot VT = RW^2 : CW \cdot WT',$$

$$\therefore \ VO \cdot VL : PV \cdot Vp = CD^2 : CP^2 = QV^2 : PV \cdot Vp,$$

$$\therefore \ VO \cdot VL = QV^2,$$

and since V is the mid-point of QQ′, therefore Q′OQL is an harmonic range. J. J. MILNE.

9. On a construction in the *Aperçu Historique*.

Many readers will have admired Chasles' elegant construction for the axes of an ellipse of which two conjugate diameters are completely given. However, the method of discovery is still more elegant and highly instructive. It is a matter of some surprise that it does not appear in didactic treatises; but Chasles was perhaps not at his best in explaining it. It appears in the notes to the *Aperçu Historique* (p. 359), and the author's object is to show that geometry need not be abandoned because some points of a figure may, under changed circumstances, become imaginary.

If CP, CD be conjugate radii of a hyperbola (and its conjugate), and if through P a parallel be drawn to CD, and along it PL, PL′ be taken each equal to CD, then we know that CL, CL′ are the asymptotes, and that the axes are the bisectors of the angle LCL′. Now the radius of the original hyperbola is not CD but CD $\sqrt{-1}$. If then in the case of the ellipse we take $PL_1 = PL_1' = CD\sqrt{-1}$, CL_1, CL_1' will be the imaginary asymptotes of the ellipse (I use the subscript $_1$ to denote imaginary points; M. Gaston Tarry, I think, suggests the use of coloured letters). But how to construct the bisectors of the angle L_1, CL_1'? A *coup de maître* solves the difficulty. The bisectors are the tangent and normal to a conic having L_1, L_1' for foci and passing through C: but this is a *real* conic, whose real foci are along a perpendicular through P to PL_1 (or CD) at distances from P equal to CD. Hence, for the ellipse, take PK, PK′ perpendicular to CD (*i.e.* along normal at P) and equal to CD; the bisectors of the angle KCK′ give the directions of the axes. No doubt Chasles was led to this construction by the then prevailing hazy impression of the geometrical interpretation of $\sqrt{-1}$.

R. W. GENESE.

10. On pairs of imaginary roots.

Elementary proof that if, in an equation

$$f(x) \equiv \ldots + p_{r+1}x^{r+1} + p_r x^r + p_{r-1}x^{r-1} + \ldots = 0,$$

p_r^2 is less than $p_{r-1}\,p_{r+1}$, the equation must have a pair of imaginary roots.

Whatever A and B may be, the equation

$$A x(p_r x - p_{r-1}) f(x) + B(p_{r+1}x - p_r) f(x) = 0$$

will be free from the term involving x^{r+1}; and if A and B be properly chosen, it will also be free from the term involving x^r. The new equation, having two consecutive terms wanting, must have at least two imaginary roots. But its roots will be the same as those of the old equation, together with those of the quadratic

$$A x(p_r x - p_{r-1}) + B(p_{r+1}x - p_r) = 0.$$

Now $x=0$ reduces the left hand side of this equation to $-p_r\mathrm{B}$, while $x=\frac{p_{r-1}}{p_r}$ gives the value $+p_r\mathrm{B}\left(\frac{p_{r+1}p_{r-1}}{p_r^2}-1\right)$. If then p_r^2 is less than $p_{r+1}p_{r-1}$, the quadratic will undergo a change of sign between the above two values of x, and therefore its roots will be real. Therefore the two imaginary roots must belong to the original equation.

This proof seems simpler than those usually given, as it only involves the most fundamental considerations. That the roots of the quadratic are real may of course be seen otherwise by applying the usual test.

PERCY J. HEAWOOD.

11. On "a solution" in No. 2.

A slip in solution No. 7 in the last issue of the *Gazette* (pp. 13, 14) should be noticed. The question proposed was to find the least velocity of a particle attached by a string to a horizontal cylinder that the string may wind itself on the cylinder without ever becoming slack. The solution assumes that the critical position is when the string extends vertically upwards; but that this is not so is seen at once by considering that the *velocity* is at this instant stationary, while the *radius* is rapidly diminishing: if the velocity were just sufficient for this position, it would be insufficient the instant before. In fact if the velocity be that due to a height h above the centre of the cylinder, radius r, l the length of the free portion of the string when horizontal, and θ its inclination in any position above the horizontal position, we have—

$$v^2=2g\,[h+r\cos\theta-(l-r\theta)\sin\theta],$$
$$\rho=l-r\theta,$$

the necessary condition being that $\frac{v^2}{\rho}-g\sin\theta$ must never be negative; *i.e.*

$$2(h+r\cos\theta)-3(l-r\theta)\sin\theta \qquad \text{(i)}$$

not negative; and its minimum value will be when

$$-2r\sin\theta+3r\sin\theta-3(l-r\theta)\cos\theta=0$$

or

$$\tan\theta+3\theta=3\frac{l}{r} \qquad \text{(ii)}.$$

Now (ii) will always have a root less than $\frac{\pi}{2}$: this substituted in (i) will determine the least possible value of h, on equating to zero.

The least initial velocity (the string being vertically downwards) will be $\sqrt{2g\left(h+l+\frac{\pi r}{2}\right)}$, with this value of h.

To solve $\tan\theta+3\theta=c$.

Let α be an approximate solution, giving $\tan\alpha+3\alpha=c+\delta$.

Then, subtracting

$$\frac{\sin(\alpha-\theta)}{\cos\alpha\cos\theta}+3(\alpha-\theta)=\delta,$$

which gives approximately for the correction $-x$ to be applied to α ($\alpha-\theta$ being small), $x(3+\sec^2\alpha)=\delta$; and so we may proceed to any required degree of approximation.

It will be found that, if α is greater than the true value of θ, so also is $\alpha-x$. P. J. HEAWOOD.

12. On the proof of the method for G.C.M.

The order adopted for proving that, with the usual method of finding a G.C.M. the last divisor is the G.C.M. required, does not seem to be the best. Thus having gone through the division process—

B)A(p
pB
C)B(q
qC
D)C(r
rD
"

the usual order is—

(i) D *divides* A *and* B.

(ii) *Whatever divides* A *and* B *divides* D.

It would be an improvement to show that *whatever divides* A *and* B *divides* C, and *whatever divides* B *and* C *divides* A.

Hence *the G.C.M. of* A *and* B *is the same as that of* B *and* C, and *by the same reasoning it must be that of* C *and* D.

Hence *if* D *divides* C *exactly it must itself be the G.C.M. of* A *and* B.

This mode of proof brings into prominence the part *each* successive remainder plays in the process.

A. LODGE.

13. On the graphic solution of quadratics.

The following problem from Leslie's *Elements* and its application to quadratic equations seems worthy of being more generally known.

To divide a straight line, whether internally or externally, so that the rectangle under its segments shall be equivalent to a given rectangle.

From the ends A and B of the straight line AB (Figs. 8, 9) erect perpendiculars AK, BL equal to the sides of the given rectangle, and in the same or in opposite directions, according as AB is to be cut internally or externally. On KL as diameter describe a circle cutting AB, or AB produced in P. Then AP, PB are the segments required.*

The demonstration may be easily seen to follow from the similarity of triangles APK, BPL or from applications of III. 35 and III. 36.

Let AB$=a$, AK$=k$, BL$=l$. Then Fig. 9 effects the solution ($x=$ AP or PB) of quadratics of the forms

$$x^2-ax+kl=0 \qquad \text{(i)},$$
$$x^2+ax+kl=0 \qquad \text{(ii)},$$

both roots being of the same sign, + in (i) − in (ii), for AP + PB = AB, AP.PB$=kl$.

Fig. 8 effects the solution of quadratics of the forms

$$x^2-ax-kl=0 \qquad \text{(i)},$$
$$x^2+ax-kl=0 \qquad \text{(ii)},$$

the roots being of opposite signs, for AP − PB = AB, AP.PB$=kl$.

* Leslie adds the following note: "The solution of this important problem now inserted in the text was suggested to me by *Mr. Thomas Carlyle*, an ingenious young mathematician, formerly my pupil."

For the application of Carlyle's solution to a problem set for Woolwich, Dec. 1894, see "Solutions," No. 20.

The solution by Pappus of the original problem is worthy of notice—

Draw AK, BL as before, let the circle on AB as diameter cut KL in y, and then the perpendicular to KL through y cuts AB in the required points (Figs. 10, 11).

All these constructions have important bearings on the geometry of conics, P being a focus of a central conic, having AB for its transverse axis and KL for a tangent. Carlyle's solution reminds us that the intercept KL made on any tangent to a central conic by the tangents at the vertex subtends a right angle at each of the foci, while that by Pappus gives the well-known property of the auxiliary circle. The figures occur in this connection in Apollonius and Gregory St. Vincent. E. M. LANGLEY.

14. On the C. G. of a circular arc.

(i) Let G_1, G_2 be the centroids of the circular arcs $CA(2\alpha)$ and $CB(2\beta)$, L and M their mid points. Then the centroid of the whole arc $BA(2\alpha+\beta)$ lies on ON, the bisector of $\angle BOA$, and it is easily seen that $\angle NOL = \beta$, $\angle NOM = \alpha$ (Fig. 12).

Hence by taking moments about ON—

$$\text{arc } CA \,.\, OG_1 \sin\beta = \text{arc } CB \,.\, OG_2 \sin\alpha,$$

$$\therefore \alpha \,.\, OG_1 \sin\beta = \beta \,.\, OG_2 \sin\alpha,$$

$$\therefore OG_1 : OG_2 = \frac{\sin\alpha}{\alpha} : \frac{\sin\beta}{\beta} \qquad (1).$$

Let β diminish indefinitely, then OG = limit of $OG_1 = r\,\dfrac{\sin\alpha}{\alpha}$.

(ii) For the sectors COA, COB we can prove equation (1) as before; and letting β diminish indefinitely G_2 becomes the C. G. of the evanescent triangle COB. Hence $OG_2 = \frac{2}{3}r$,

and $$OG = \tfrac{2}{3}r\,\frac{\sin\alpha}{\alpha}.$$

(iii) For the surfaces or volumes of lunes of a sphere bounded by secondaries of the great circle ACB, we can prove equation (1) as before, and putting $2\beta = \pi$ the lune (2β) becomes a hemisphere.

Hence *for surface* $OG_2 = \frac{1}{2}r$,

$$\therefore OG_1 = \tfrac{1}{2}r\,\frac{\sin\alpha}{\alpha} \cdot \frac{\frac{1}{2}\pi}{\sin\frac{1}{2}\pi}$$

$$= \frac{r}{4\pi} \cdot \frac{\sin\alpha}{\alpha};$$

while *for volume* $OG_2 = \frac{3}{8}r$,

$$\therefore OG_1 = \frac{3r}{16\pi} \cdot \frac{\sin\alpha}{\alpha}.$$ G. H. BRYAN.

15. On another proof of XI. 4.

Let CO be perpendicular to OA, OB and OP any line through O in the plane AOB. Draw AB perpr. to OA.

Square on CB = squares on CO, OB = squares on CO, OA, AB = squares on CA, AB; therefore $\angle CAB$ is right.

Therefore square on CP = squares on CA, AP = squares on CO, OA, AP = squares on CO, OP, therefore $\angle COP$ is right. W. GALLATLY.

16. On the centroid of a trapezoid.

As in the investigation in No. 3 the parallel sides AB, DC of the trapezoid ABCD being supposed to contain p and q units of length respectively, the triangles ABD, BDC may be regarded as containing $3p$ and $3q$ units of mass respectively. Hence the centroid of the trapezoid is that of the system of particles containing p units each at A,B,D and q units each at B,D,C—*i.e.* of the system of 4 at A,B,C,D containing p, $p+q$, q, $p+q$ units respectively. Similarly it must be that of system of 4 at A,B,C,D containing $p+q$, p, $p+q$, q units respectively. Hence, superposing the two systems it must be that of $2p+q$ at A, $2p+q$ at B, $p+2q$ at C, $p+2q$ at D, and therefore of $2(2p+q)$ at E and $2(p+2q)$ at F. D. QUINT.

17. On the theorem *ad quatuor lineas.*

It may be worth noticing that the value of k obtained in Note 4 (No. 3) comes at once for *the ellipse* by projection thus: If α, β, γ, δ be the perpendiculars from any point P on a circle upon the sides of an inscribed quadrilateral ABCD, we have by elementary geometry $\alpha\gamma = \beta\delta$ or $\alpha : \beta :: \delta : \gamma$.

If then the sides of the quadrilateral be produced so that $AA' = BB' = CC' = DD' =$ diameter of circle, we have

$$\Delta OAA' : \Delta OBB' :: \Delta ODD' : \Delta OCC' \qquad (i)$$

(their bases being equal).

Now let the circle be projected into an ellipse; (i) will still hold, and we deduce at once—

$$\alpha \,.\, Oa : \beta \,.\, Ob :: \delta \,.\, Od : \gamma \,.\, Oc$$

(since the bases of the triangles will still be equal to the *parallel* diameters $2Oa$, $2Ob$, $2Oc$, $2Od$), *i.e.*

$$\frac{\alpha\gamma}{\beta\delta} = \frac{Ob \,.\, Od}{Oa \,.\, Oc}.$$

PERCY J. HEAWOOD.

SOLUTIONS OF EXAMINATION QUESTIONS.

The Editor will be glad to avail himself of the help of all classes of readers towards making this section of the Gazette *as useful as possible.* MATHEMATICAL TUTORS *are invited to send neat solutions;* STUDENTS *to call attention to classes of problems presenting exceptional difficulties, and* EXAMINERS *who sympathise with us to forward copies of their papers. The help of foreign readers is especially requested in obtaining copies of papers set in the public examinations of other countries.*

The Editor acknowledges with thanks the receipt of the following sets of papers:—

University of Texas (Dec. 1894, Modern Geometry).

18. *If θ be the acute angle between a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the focal distance of the point, prove that the distance of the point from the centre is $\sqrt{a^2 - b^2\cot^2\theta}$.* [W. 1881.]

With the usual notation we have

$$b^2 = SY \cdot S'Y' = SP \cdot S'P \sin^2\theta = CD^2 \sin^2\theta,$$

since $SP \cdot S'P = CD^2$, therefore $CD^2 = b^2 \operatorname{cosec}^2\theta$.

Now $CP^2 = a^2 + b^2 - CD^2$

$= a^2 - b^2 (\operatorname{cosec}^2 \theta - 1),$

$\therefore CP = \sqrt{a^2 - b^2 \cot^2 \theta}.$

H. A. HALL.

19. *Find coefficient of xyz in expansion of* $\{(1-x-y-z)(1-x-y-z+4xyz)\}^{-\frac{1}{2}}$ *as a rational integral function of x, y, and z.* [W. Dec. 1894.]

Expansion

$$= (1-x-y-z)^{-1} \times \left(1 + \frac{4xyz}{1-x-y-z}\right)^{-\frac{1}{2}}$$

$$= \{1 + (x+y+z) + (x+y+z)^2 + (x+y+z)^3 + \ldots\}$$

$$\times \left\{1 - \frac{1}{2} \cdot \frac{4xyz}{1-(x+y+z)} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2} \cdot \frac{4x^2y^2z^2}{1-x-y-z} + \ldots\right\}$$

$$= \{1 + x+y+z + x+y+z^2 + (x+y+z)^3 + \ldots\}$$

$$\times \{1 - 2xyz\,(1 + (x+y+z) + \ldots) + \text{higher powers}\}.$$

Thus term containing xyz is $6xyz - 2xyz$, or $4xyz$.

20. Describe a circle passing through two given points and intercepting, on a given line, a segment of a given length. [W. Dec. 1894.]

(i) Let C,D, the given points, be on the same side of the given line; let AB be the given segment, E its mid point. Produce CD to cut the given line in O. Draw FG($= \frac{1}{2}$ AB) perp. to OF (a mean propl. between OC, OD). Make OH = OG, mark off HT, HS each equal to BE (Fig. 7).

$$OG^2 = OF^2 + FG^2 = OD, OC + BE^2.$$

But $OG^2 = OH^2$

$= HT^2 + OT \cdot OS = OT \cdot OS + BE^2,$

$\therefore OD \cdot OC = OT \cdot OS,$

therefore C,D,S,T are concyclic.

The solution requires some modification when the given points C,D are on opposite sides of the given line.

J. H. HOOKER.

[This problem, when the given points are on the same side of the given line, depends for its solution on the same construction as the following:—*To produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced shall be equal to the rectangle contained by two given straight lines* (See T. Simpson's *Elements of Geometry*, ed. 1821, p. 103). When this is investigated algebraically it gives rise to the same equation $x^2 + ax = kl$ as the above. The solution $\left(x = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + kl}\right)$ leads directly to Mr. Hooker's construction. Another less obvious solution from Leslie, *due to T. Carlyle* (see Note 13), is added on account of the interest attaching to it. It has the advantage of dealing with both cases at once. Let CD meet the given line in O. Draw AK, BL perpendicular to AB and equal to CO, OD. AK, BL must be drawn on the same or on the opposite side of AB, according as CO, OD are in the same sense or in opposite senses. Let the circle on KL as diameter cut AB in P. From O along the given line cut off OS, equal to PA. The circle through S, C, D shall be the one required. For let circle CSD cut the line again in T, then

$OS \cdot OT = OC \cdot OD = AK \cdot BL = PA \cdot PB$ (see Note 13),

$\therefore OT = PB,$

$\therefore ST = AB.$

Note that the quadratic equation for the case in which C and D are on opposite sides of the line is $x^2 - ax = -kl$, and that it has two positive solutions $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - kl}$. The solution is of course impossible if $kl > \frac{a^2}{4}$, and accordingly the geometrical construction is seen to fail when $CO \cdot OD$ is greater than the square on AE. EDITOR.]

21. *The longest side of a triangular plot of ground is* 100 *yards, its perimeter is* 250, *and one angle is* 40°. *Find the remaining angles.* [W. Dec. 1894.]

40° cannot be greatest angle, and therefore is not opposite the longest side (100). (i) Make AB 100. Angle BAE = 40°. Make AE = 150. Join EB and make EBC = CEB. Therefore ABC is Δ.

Then in Δ ABE—

$$\left.\begin{matrix} AE = 150 \\ AB = 100 \end{matrix}\right\} \text{ included angle } A = 40°.$$

The angle E can be found, and C = 2E, therefore angle ABC = 180 − C − A. J. H. HOOKER.

(ii) As in (i) 100 is one of the sides (AB) adjacent to 40°, which is taken as A (Fig. 13). Produce AB to N, making AN = 125. Then N is the point of contact with AB produced of the excircle opposite A. Draw AO, making angle NAO = 20° and NO perpendicular to AN, then O is the centre of this circle.

Hence $\tan \frac{1}{2}B = \tan BON$

$$= \frac{25}{125 \tan 20°}$$

$$= \tfrac{1}{5} \tan 70° \qquad \cdot 438934$$

$$\therefore \tfrac{1}{2}B = 28°47'21'' \qquad \bar{1}\cdot 301030$$

$$B = 57°34'42'' \qquad \overline{\qquad\qquad}$$

$$C = 82°25'18'' \qquad \bar{1}\cdot 739964$$

The student's attention is drawn to the general problem. *Given an angle, one of the sides containing it and the perimeter of a triangle, to construe it.*

Draw ABN equal to the semiperimeter and mark off AB equal to the given side, and angle NAO equal to half the given angle A. Draw NO perpendicular to AN. Then the triangle is completed by drawing in the other two tangents from A and B to the circle whose centre is O and radius ON. Compare the *second* solution of problem 1 on p. 5 of No. I of the *Gazette*, which depends on the same general theorem as the above. EDITOR.

22. If $r = p/(1 + e\cos\phi)$,

$r' = p/(1 + e\cos\phi')$,

$r'' = p/(1 + e\cos\phi'')$,

express $u = \dfrac{\sin(\phi'' - \phi')}{r} + \dfrac{\sin(\phi - \phi'')}{r'} + \dfrac{\sin(\phi' - \phi)}{r}$

in a form adapted to logarithmic computation and evaluate it when $\phi = 17°4'$, $\phi' = 22°27'$, $\phi'' = 38°19'$, $p = 21$.

[W. Dec. 1894.]

$pu = (1+e\cos\phi)\sin(\phi''-\phi') + (1-e\cos\phi')\sin(\phi-\phi'') + \ldots$
$= \sin(\phi''-\phi') + \sin(\phi-\phi'') + \sin(\phi'-\phi)$
$+ e\{\cos\phi\sin(\phi''-\phi') + \cos\phi'\sin(\phi-\phi'') + \cos\phi''(\sin\phi'-\phi)\}.$

Now the coefficient of e on expansion vanishes,

$$\therefore pu = 2\sin\frac{\phi-\phi'}{2}\cos\frac{2\phi''-\phi'-\phi}{2} + 2\sin\frac{\phi'-\phi}{2}\cos\frac{\phi'-\phi}{2}$$

$$= 2\sin\frac{\phi'-\phi}{2}\left\{\cos\frac{\phi'-\phi}{2} - \cos\frac{2\phi''-\phi'-\phi}{2}\right\}$$

$$= 4\sin\frac{\phi'-\phi}{2}\sin\frac{\phi''-\phi}{2}\sin\frac{\phi''-\phi'}{2},$$

$$u = \frac{4}{21}\sin 2°36'30'', \sin 10°37'30'', \sin 8°1'.$$

$\log 4 = \cdot 602060$
$\operatorname{colog} 21 = \bar{2}\cdot 677781$
$\log \sin 2°36'30'' = \bar{2}\cdot 658090$
$\log \sin 10°37'30'' = \bar{1}\cdot 265714$
$\log \sin 8°1' = \bar{1}\cdot 144453$
$\log u = \bar{4}\cdot 348098$
$u = \cdot 00022289$

J. H. HOOKER.

[It is noticeable here that as the coefficient of e disappears no advantage is gained by the use of logarithms. If a table of natural sines to a sufficient number of places is available—

$21u = \sin 16°2' + \sin 5°13' - \sin 21°15'$ ·2761965
·0909223
·3671188
$u = \cdot 0002229$ ·3624380
21)·0046808
468
1
·0002229

EDITOR.]

23. *From the point* O *the three straight lines* OA, OB, OC *are drawn in the same plane, of lengths* 1, 2, 3 *respectively, and with the angles* AOB, BOC *each equal to* 60°. *Find the angle* ABC. [W. Dec. 1894.]

On OC mark off OD = 1. Then (Fig. 14) each of triangles AOB, DOB is half an equilateral triangle.

Hence $\angle ABD = 60°$. 1·73205
·57735
and $DB = \sqrt{3}$. 1·15470
43
Hence $\tan DBC = \frac{2}{\sqrt{3}}$ 27
68)1620
$\angle DBC = 49°6'23''\cdot 8$ 260
56
$\angle ABC = 109°6'23''\cdot 8$. 23″·8

EDITOR.

24. *Four circular coins of different sizes are placed upon a table so that each one touches two and only two of the remaining three. Show that the four points of contact lie on a circle.*

Let A, B, C, D be the points of contact of the four coins APB, BQC, CRD, DSA; E, F, G, H their centres (Fig. 15). Let the angle DAB be divided into the two angles DAT, TAB by the common tangent AT at A.

Then $\angle TAB = \angle$ in alt. segt. APB
$= \frac{1}{2}\angle E$.
Similarly $\angle DAT = \frac{1}{2}\angle H$
$\therefore \angle DAB = \frac{1}{2}(E+H)$.
Similarly $\angle BCD = \frac{1}{2}(F+G)$
$\therefore \angle^s DAB, BCD =$ two rt. $\angle^s$.

Note (i) *that* AB, BC, CD, DA *are parallel to the external bisectors of* E, F, G, H. Hence we are reminded of the theorm that *the four straight lines bisecting the exterior angles of any quadrilateral form a cyclic quadrilateral.*

(ii) *That* EF + GH = FG + HE.

Hence a circle can be inscribed in the quadrilateral EFGH. It can easily be shown that this has the same centre as the circle round ABCD, though it does not in general coincide with it. EDITOR.

25. *Two tangents* OP, OQ *are drawn from* O (hk) *to the ellipse* $\frac{(x-p^2)}{a^2} + \frac{(y-q)^2}{b^2} = 1,$ *whose centre is* C. *Show that the area of the quadrilateral* OPCQ *is* $\sqrt{b^2(h-p)^2 + a^2(k-q)^2 - a^2b^2}$.

[Clare and Trin. Hall, Jan. 1895.]

First transfer to a new origin at the point (pq) without altering directions of axes, then the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and it will be sufficient to prove that the area of the quadrilateral is $\sqrt{b^2h^2 + a^2k^2 - a^2b^2}$.

Let P be the point $a\cos\phi$, $b\sin\phi$.

Let Δ represent the area of the triangle CPO, then area of quadrilateral

$$= 2\Delta = ak\cos\phi - bh\sin\phi \qquad (1)$$

also co-ordinates of P satisfy equivalents to PQ,

$$\therefore \frac{h\cos\phi}{a} + \frac{k\sin\phi}{b} = 1 \qquad (2).$$

Thus $bh\cos\phi + ak\sin\phi = ab,$
$ak\cos\phi - bh\sin\phi = 2\Delta;$

square and add to eliminate ϕ, then

$$4\Delta^2 = a^2k^2 + b^2h^2 - a^2b^2$$

or $$2\Delta = \sqrt{a^2k^2 + b^2h^2 - a^2b^2}.$$

T. D. DAVIES.

26. *Given that the two quadratic factors* $a_1x^2 + 2h_1x + b_1$ *and* $a_2x^2 + 2h_2x + b_2$ *have a common factor; prove by any method the equation of condition*

$$(a_2b_1 + a_1b_2 - 2h_1h_2)^2 = 4(h_1^2 - a_1b_1)(h_2^2 - a_2b_2).$$

[Intermediate in Arts.]

This equation may, by simplification and reduction, be easily shown equivalent to the more commonly given

$$(a_1b_2 - a_2b_1)^2 = 4(a_1h_2 - a_2h_1)(h_1b_2 - h_2b_1).$$

But it is interesting to see how the required condition may be directly obtained.

By hypothesis $a_1x^2 + 2h_1x + b_1 = (px + q)(lx + m)$,

$a_2x^2 + 2h_2x + b_2 = (rx + s)(lx + m)$,

$\therefore (a_1 + \lambda a_2)x^2 + 2(h_1 + \lambda h_2)x + b_1 + \lambda b_2 = \{(p + \lambda r)x + q + \lambda s\}(lx + m)$.

Hence the left-hand expression will or will not be a perfect square according as $\frac{p + \lambda r}{l}$ is or is not equal to $\frac{q + \lambda s}{m}$, *i.e.* according as $\lambda(mr - ls)$ is or is not equal to $lq - mp$, which gives one and only one real value for λ.

But the expression will be a perfect square if—

$$(h_1 + \lambda h_2)^2 = (a_1 + \lambda a_2)(b_1 + \lambda b_2),$$

i.e. if

$$\lambda^2(h_2^2 - a_2b_2) - \lambda(a_1b_2 + a_2b_1 - 2h_1h_2) + h_1^2 - a_1b_1 = 0,$$

and this quadratic will have two different roots unless $(a_1b_2 + a_2b_1 - 2h_1h_2)^2 = 4(h_2^2 - a_2b_2)(h_1^2 - a_1b_1)$.

For another solution, see Steggall's *London University Questions and Solutions* (Van Voorst).

The student will perhaps be reminded of the important theorem in analytical geometry, that if $u = 0$, $v = 0$ be any pair of straight lines; any other straight line through their point of intersection can be represented by an equation of the form $u + \lambda v = 0$.

Thus, if $px + qy = 0$, $rx + sy = 0$, $lx + my = 0$ be any three straight lines through the origin, the equation to $lx + my = 0$ may be expressed in the form $px + qy + \lambda(rx + sy) = 0$.

The ordinary form of the same condition, viz.—

$$(a_2b_1 - a_1b_2)^2 = 4(a_1h_2 - a_2h_1)(h_1b_2 - h_2b_1)$$

may, as the Rev. J. J. Milne has pointed out, be immediately obtained from the consideration that the *equations*

$$a_1\lambda^2 + 2h_1\lambda + b_1 = 0,$$
$$a_2\lambda^2 + 2h_2\lambda + b_2 = 0$$

must be simultaneously satisfied; hence

$$\frac{\lambda^2}{h_1b_2 - h_2b_1} = \frac{2\lambda}{a_2b_1 - a_1b_2} = \frac{1}{a_1h_2 - a_2h_1},$$
$$\therefore (a_2b_1 - a_1b_2)^2 = 4(a_1h_2 - a_2h_1)(h_1b_2 - h_2b_1).$$

The solution given accounts for the *other form* of the condition. EDITOR.

27. *Solve the equations*—

$$(x \cos \alpha - c) \sin(\alpha + \beta) = (y \cos \alpha - c) \sin(\alpha - \beta) \quad \text{(i)},$$
$$x \cos(\alpha + \beta) = y \cos(\alpha - \beta) \quad \text{(ii)}.$$

Show that—

$$x + y = 2c \cos \alpha \quad \text{(iii)}.$$

[W. 91.]

As the equations are merely simultaneous and linear we could of course eliminate one letter in the ordinary way, but the rider suggests the use of an indeterminate multiplier, as used in analytical geometry, and referred to in solution 26.

In (ii) multiply by λ and then add—

$$x(\cos \alpha \sin \overline{\alpha + \beta} + \lambda \cos \overline{\alpha + \beta}) - y(\cos \alpha \sin \overline{\alpha - \beta} + \lambda \cos \overline{\alpha - \beta}) = c(\sin \overline{\alpha + \beta} - \sin \overline{\alpha - \beta}) \quad \text{(iv)}.$$

If this is to reduce to (iii) we must have the coefficients of x and y equal, and hence—

$$\lambda(\cos \overline{\alpha + \beta} + \cos \overline{\alpha - \beta}) = -\cos \alpha(\sin \overline{\alpha - \beta} + \sin \overline{\alpha + \beta}),$$
$$\lambda = -\sin \alpha.$$

On substitution (iv) reduces to (iii), and eliminating between (ii) and (iii) we obtain—

$$x \cos \beta = c \cos(\alpha - \beta), \quad y \cos \beta = c \cos(\alpha + \beta).$$

EDITOR.

QUESTIONS FOR SOLUTION BY STUDENTS.

40. Prove that if Y be the foot of the perpendicular from S on the tangent at the point P of an ellipse, the angle PSC is double of the angle SYA. P. J. HEAWOOD.

41. Prove that if a parabola and an ellipse have a common latus rectum the intercept on the axis major between the focus and a common tangent will be bisected at a vertex of the ellipse. P. J. HEAWOOD.

42. I measure a field by pacing. I then make a sketch of the land, using a scale of paces which is on the scale of 8 inches to a mile. The regulation pace is 32 inches. I find after drawing my sketch that my pace is only 31 inches. What should be the length of a special scale of yards for the sketch, to show 1000 yards in all?

J. H. HOOKER.

43. A bullet is fired and hits a wooden target horizontally, and just passes through it when the target is 1 foot thick. When the target is 9 inches thick the bullet fired under same conditions afterwards strikes the horizontal ground 400 yards farther on. How far from the firing point is the target. J. H. HOOKER.

44. BE, CF are perpendiculars from BC to the opposite sides CA, AB of triangle ABC; EG, FH are perpendiculars on CF, BE respectively. Prove that EF is a mean proportional between BC, GH. R. TUCKER.

45. ABC is any triangle; P any point. Show that it is possible to construct an equilateral triangle DEF such that PD, PE, PF are equal to BC, CA, AB respectively. Show also that the square on one of its sides $= \frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\,\Delta$. T. H. DE LAND.

BOOKS AND MAGAZINES RECEIVED.

A First Step in Euclid. By J. G. BRADSHAW. (Macmillan & Co.)

A Class in Geometry. By GEORGE ILES. (Kellogg & Co., New York.)

Un'Applicazione della Teoria dei Numeri alle Frazioni Decimali Periodiche. By B. CARRARA.

The American Mathematical Monthly. November and December 1894.

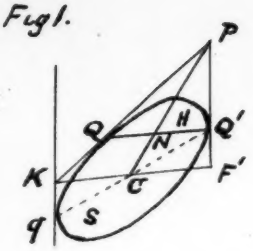
Journal de Mathématiques Élémentaires. December 1894, January and February 1895.

Periodico di Matematica. November-December 1894.

All inquiries should be addressed to the Editor of the Mathematical Gazette, 16 *Adelaide Square, Bedford.*

The Mathematical Gazette No. 4.

Fig 1.



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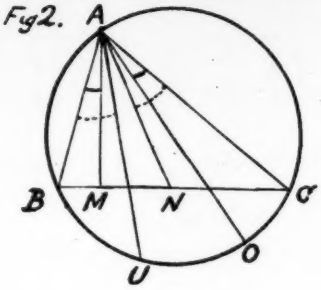

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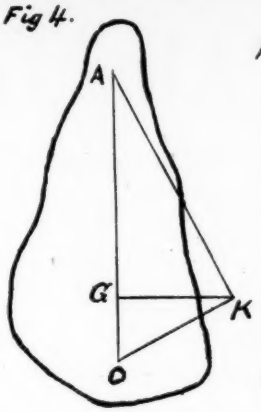

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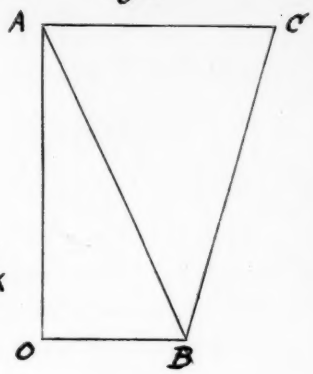

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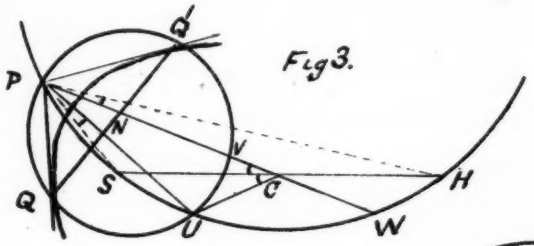

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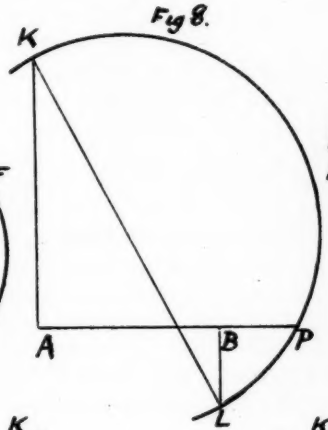

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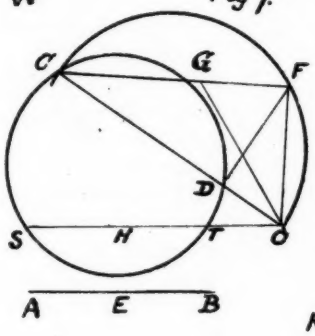

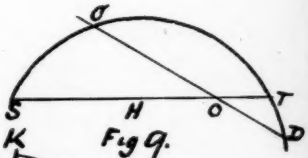

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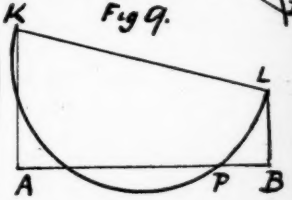

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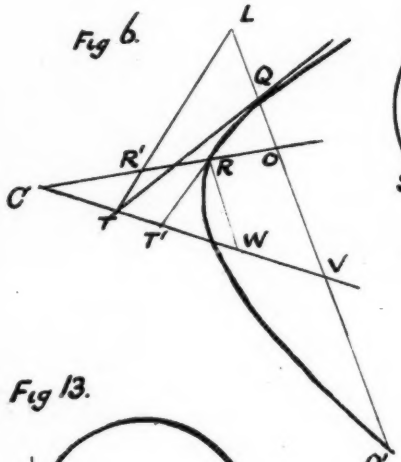

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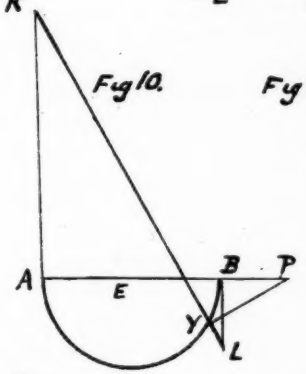

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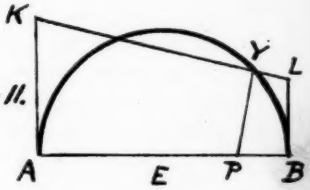

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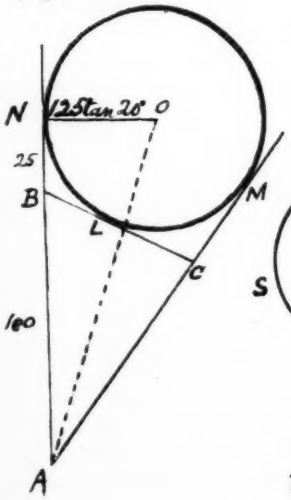

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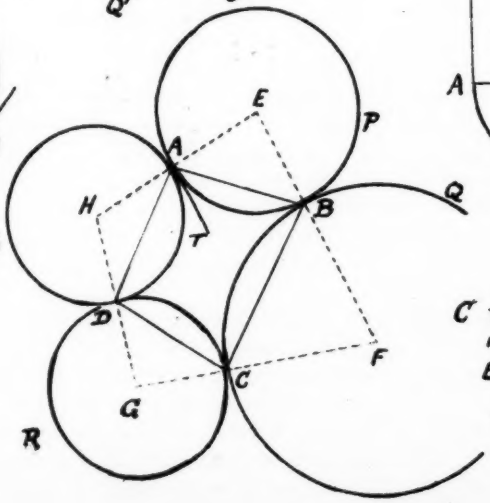

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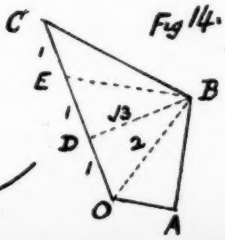
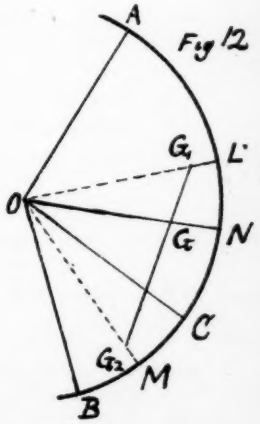
Fig 12.